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A New Approach to Multi Objective Fuzzy Fractional Linear Programming

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Abstract

The field of application is Iraqi Light Industries Company, the best items that must be preserved were chosen to maximize profit at low cost, and also we are trying to get one of the direct and effective methods that contain some arithmetic operations to obtain the optimal real values. First, the paper will describe the data and create the mathematical model for the Multi-Objective Fuzzy Linear Programming Problem (MOFLFPP) relevant to the study problem. In light of production process restrictions that may limit the company's ability to provide products in the right quantity and time, the second section focuses on solving the model and finding the optimal solution, which is the production mix that maximizes profits at the lowest cost. A MOFLFPP is transformed into a Linear Programming Problem (LPP) through the use of α-cut and Max-Min technique. In order to demonstrate the effectiveness of the strategy that we have proposed, a sample problem from real life has been used. Decision-makers will be better able to appreciate the value of the MOFLFPP if they are given the opportunity to discuss the practical challenge. A result analysis is also constructed for the purpose of determining whether or not our method is applicable, and how it compares to other ways. The strategy worked and yielded solid results from extreme point that provided an optimal answer.

Keywords: Fuzzy sets, Linear programming, Linear fractional programming, Fuzzy coefficients.

1 | Introduction

The need for computerized scientific research arose as a result of the quick advancements in science at the level of educational, manufacturing, and service institutions, particularly in operations research, where it is necessary to use computerized applications to study program changes that result from significant changes in plans and programs. The majority of the challenges faced by decision-makers arise when there are multiple goals to accomplish a given problem, which is one of the most significant objectives of operations research

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and hence ranks first in decision-making, and when we encounter a situation where two goals are incompatible, one of them is relative to the other goal, requiring that one be maximized and the other be minimized. Herein lies the significance of Linear Fractional Programming (LFP) and non-LFP, which are regarded as two of the most important topics for solving such cases and have a wide range of applications in this field. They aim to maximize various factors, such as production relative to the workforce, production relative to costs, production relative to manufacturing process waste, and others.

Numerous fields have used fuzzy sets theory, including mathematical modeling, industrial applications, control theory, management sciences, and more. When these kinds of real-world issues are transformed into mathematical programming problems, we typically run into some challenges. Uncertainty in Decision Makers' (DM) preferences, knowledge, and/or information is one of the challenges. If the coefficients of the model are correctly determined as real numbers, then the problem formulation is either unsolvable or produces an incorrect solution. In these situations, we must look into the reasons, adjust the real numbers that have been determined, answer the reformulated problem, and then continue the process until we have a workable solution.

Fuzzy programming approach does not require a large amount of money to acquire the correct coefficient value or distribution, unlike classical and stochastic programming approaches might. This means that in situations when human expertise is able to specify coefficients only roughly rather than precisely, the fuzzy programming approach will prove to be highly beneficial.

The fact that many manufacturing companies suffer large losses due to high production costs and low earnings serves as evidence of the research problem. These losses can be attributed to a number of factors, such as the devastation caused by war, the inability of governments to provide financial support in the form of long-term, soft loans, or their inaction customer protection, domestic product protection, customs, the high cost of raw materials, and the public's unwillingness to buy local products in favor of imported goods with more current specifications and lower prices because there are no taxes on imported goods entering the country. Furthermore, a lot of product damage occurs as a result of power outages, which raises expenses for the company. Furthermore, because certain materials are scarce due to outdated molds and machine and equipment designs, the company contacts foreign manufacturers to produce them and sell them at premium prices. Every one of these is a paid matter. The corporation endeavors to manufacture the specified quantity of goods, taking into account the amount of raw materials available and the extent of the product's demand. It therefore seeks the optimal production volume at the lowest feasible cost and a fairly high profit rate.

The research is important because of the following:

I. It is an attempt to investigate whether there are obstacles preventing the Light Industries Company from using the Fuzzy Linear Fractional Programming (FLFP) problem and the opportunities available for its application.

II. Resolving the company's problems involves selecting the best plan of action and determining how many units to manufacture from the production mix in order to maximize the ratio of profits to costs.

The main objective of the paper using the FLFP problem to determine the optimal number of products that achieve the maximum profit relative to the lowest cost through the application of the Max-Min method, the research aims to develop a mathematical model to maximize the ratio of returns on costs for the production company.

Several academics have studied FLFP. The investigator gave Das et al. [1] a study in which a novel ranking function was constructed to address Fully FLFP. Nonlinear functions are used to substitute the non-parallel sides of the trapezoidal fuzzy number to obtain the ranking function. Numerous numerical examples are shown and contrasted with the previous techniques. It was out that using his approach to solve the FFLFP problem is more convenient and straightforward. Das et al. [1] suggested a LFP model with absolute value functions. The main contribution is transforming LFP into linear programming issues with specific theorems and solving them with a famous approach, and also used simplex methods to solve an absolute value linear

programming issue optimally. Additionally, they compare this procedure to another. They also provide numerical tests to support their claims and admit that the novel transformation method may offer useful tools for solving absolute LFP problems. The numerical findings show that the new method outperforms the others. With enough processing time, the methods may handle enormous problems. Gurmu and Fikadu [2] covered fuzzy programming and Bi-Level Linear Programming (BLPP) applications.

They also used BLPP and Fuzzy Mathematical Programming (FMP) to solve the issue. They propose the FMP approach for linear membership function-based objective minimization. The higher DM supervises FMP. The upper-level decision-maker specifies the preferred values of decision variables under his control to help the lower-level DM find his optimum in a wider feasible space and the bounds of his objective function to guide his solution search.Even if the decision-making process is from higher to lower, the lowest level is most crucial. The decision vector under lower-level DM control has no tolerance restrictions. Thus, this decision vector remains unaltered or approaches its isolated value. At a higher level, decision vectors are given a tolerance and can move within it. Individual tolerance levels can be regarded variables, and DM cooperation can optimize the entire system. We can simply apply this to non-linear BLPP. Das et al. [1] defined a LFP problem with triangular fuzzy costs and constraints. They used the centroid ranking function to transform the fuzzy LFP problem into a Crisp Line Fractional Programming (CLFP) problem. they proposed solution is based on crisp LFP and has a basic structure. To demonstrate its efficiency, they demonstrated a real-life scenario and used centroid ranking function to overcome all of its constraints. Finally, their method's applicability is assessed by result analysis. Das and Edalatpanah [3] extended LFP issues to Neutrosophic Sets (NSs) and examined their operations and usefulness. The novel approach uses triangular neutrosophic set arithmetic and aggregation ranking function. In this study, they tackle a neutrosophic triangular fuzzy number problem with equality and inequality constraints for the first time. Some numerical models are explored to assess our technique's legitimacy, profitability, and materiality based on real issues. Finally, some numerical trials and one contextual analysis demonstrate that the novel methods are better than current ones. Borza et al. [4] propose a MOLLFPP solution. The strategy involves two phases. Phase 1 creates a variable transformation-based global optimal solution of the Linear plus Linear Fractional Programming Problem (LLFPP). This step converts LLFPP to linear programming. Phase 2 applies weighted sum and max–min algorithms to translate MOLLFPP into LPPs using phase 1 data. The method is demonstrated by solving two problems and comparing their accuracy and also The outcomes showed that the suggested strategy performed accurately and efficiently.

2 | Linear Fractional Programming

With the growth of production facilities and the diversification of their activities, numerous challenges and variables have emerged, impacting, in one way or another, the ability to make the right decisions. This necessitates new approaches that assist in making several critical decisions faced by senior management, and given their wide range of applications in information theory, applied linear algebra, large-scale programming, game theory, transportation, production, finance, location theory, stochastic processes, Markov renewal programs, and resource allocation, LFP problems are highly interesting. In numerous real-world scenarios, optimizing ratios of criteria provides greater context-specific understanding than optimizing each criterion separately [5], [6].

This is a special case of non-linear programming, where the objective function is a ratio between two fractional objective functions. The numerator represents a function intended for maximization, while the denominator represents a function intended for minimization (in the case of the objective function is Max). However, in the case of the objective function Min, the numerator function is intended for minimization, and the denominator function is intended for maximization. The relationship between the variables in both the numerator and denominator of the objective functions must be linear, subject to constraints that also have linear relationships between their variables. If the fractional function is Min, the numerator represents a linear objective function to be minimized, while the denominator represents a linear objective function to be maximized, also constrained linearly.

According to [7]–[12], this section will cover the general form of the LFP that we will enforce, which is as follows:

$$\text{Maximize}\left\{(Z)\frac{C^tX+\theta_1}{D^tX+\theta_2}\right\}. \tag{1}$$

s.t $AX \le b, D^tX+\theta_2 > 0, X \ge 0,\ X, C^t, D^t \in R^n,\ A \in R^{mxn},\ \theta_1, \theta_2 \in R,$

where

C^t : coefficients of decision variables in the numerator objective function.

D^t : coefficients of decision variables in the denominator objective function.

X : a vector representing n-dimensional decision variables.

A : matrix of degree

b : a vector representing the available resources for constraints.

θ_1, θ_2 : is constants.

Following Charnes and Cooper's approach *Eq. (1)* is transformed into the following linear Problem utilizing variable transformations, by [6]–[9], [13]

$$\lambda = \frac{1}{D^tX+\theta_2},\ Y = \lambda X,$$

$$\text{Maximize}\ \left\{C^tY+\theta_1(\lambda)\right\}, \tag{2}$$

$$AY - Nb(\lambda) \ge 0,\ \ D^tY+\theta_2(\lambda) \le 1,\ \ Y, \lambda \ge 0.$$

Theorem 1. The optimal solution for *Eq. (2)* by imposition $(Y^{(*)}, \lambda^{(*)})$, is based on that the $X^{(*)} = \left(\frac{Y^{(*)}}{\lambda^{(*)}}\right)$ an optimal solution to *Eq. (1)*.

3 | Multi-Objective LFP Problem

3.1 | Regarding the General form of the Multi-Objective Programming Problem

The following phases are usually included in the process of solving a MOLFP problem [6], [10], [13], [14]:

I. Formulating the problem: identify the constraints, objective functions (fractional), and decision variables.

II. Normalization (optional): apply methods such as the Charnes-Cooper transformation to translate the fractional objectives into a linear form.

III. Weighting or scalarization: apply the ε -constraint method to reduce the multi-objective problem to a single-objective problem.

IV. Scalarized problem solution: apply strategies such as the Simplex approach to solve the resulting Linear Programming (LP) problem.

V. Optimality pareto verify whether the result is pareto-optimal, which states that no goal can be advanced without making at least one other goal poorer.

VI. Iteration (if necessary): if the answer is not satisfactory, repeat the procedure after adjusting the weights or constraints.

VII. Making decisions: examine the outcomes and select the best solution based on decision-maker preferences.

Let us take into consideration the general form of the multi-objective LFP problem is as follows:

$$\text{Maximize}\{F_1(X), F_2(x), F_3(X), \ldots\ldots\ldots\ldots, F_n(X)\},$$
$$\text{s.t } X \in S,$$
$$F_n(X) = \frac{C^t X + \theta_1}{D^t X + \theta_2} \quad \text{where } X, C^t, D^t \in R^n \text{ and } \theta_1, \theta_2 \in R. \tag{3}$$

Definition 1 ([15]). $X^{(*)} \in S$ is a solution and should be called efficient if and only if $X \in S$ is not exists such that $F_j(X^{(*)}) \le F_j(X), j = 1, 2, 3, \ldots, k$ and $l \in \{1, 2, \ldots, k\}$ is exists such that $F_j(X^{(*)}) < F_j(X)$. The (Max-Min) approach is a traditional method that is can be used to secularize the Multi-Objective Programming Problem (MOPP) as follows:

$$\text{Maximize}(\beta) \text{ s.t } X \in S, \ \beta \le F_i(X) \text{ for i=1,2,..., k } \theta_1, \theta_2 \in R. \tag{4}$$

Definition 2 ([13]). Let's think a little bit about the problem of a single-objective $\underset{X \in S}{\text{Maximize}} \ g(X)$. If $g(X) \le g(X^*) + \varepsilon, \text{for all } X \in S$. It can be said that the point $X^* \in S$ is $\varepsilon-$optimal solution.

4 | Problem Definition and Preliminaries

Before we delve into the analysis, let's acquaint ourselves with certain concepts and notions concerning fuzzy numbers [6], [7], [12], [13], [16]–[19]:

4.1 | Fuzzy Sets

This section aims to recapitulate the fundamental definitions and findings on fuzzy sets and related subjects. Additionally, we offer a succinct yet fundamental explanation of fuzzy arithmetic and fuzzy integers. We will talk more about ranking of fuzzy numbers in following chapters since it is a crucial topic in the study of FMP. and describe it mathematically as follows:

Definition 3. In set universe X There is a group named is $(\tilde{A})$ such that

$$\tilde{A} = \{(x, \tilde{A}(x)) : x \in X\}, \tag{5}$$

where $\tilde{A}(x)$ is the membership degree from point x , and is a real number in the interval [0, 1]. The range in which this function is specified is [0, 1], $\tilde{A}(\bullet)$, or

$$\begin{aligned} &\tilde{A}(\bullet) : X \to [0,1], \\ &x \to \tilde{A}(x), \end{aligned} \tag{6}$$

as a membership function for a fuzzy set $(\tilde{A})$.

The fuzzy sets' *Definition 3* leads to the following few conclusions:

I. Classical sets have been expanded upon by the idea of fuzzy sets.

If $f(X)$ let us say that is the fuzzy set for x i.e. $C(X) \subset f(X)$, If the fuzzy membership function $(\tilde{A})$ takes only two values, 0 or 1, we will call it a classical set of x.

II. The characteristic function idea is expanded upon by the membership function concept.

When $A \in C(X)$ is a regular subset within X, then A's characteristic function is

$$\mu_A(x) = \begin{Bmatrix} 1, x \in A(\text{membership degree of x for A is 1} \\ 0, x \notin A(\text{membership degree of x for A is 0} \end{Bmatrix}. \tag{7}$$

In other words, in fuzzy sets, x belongs to degree $(\tilde{A})$ more strongly when the membership degree $\tilde{A}(x)$ in the fuzzy set $(\tilde{A})$ is closer to 1 than when it is closer to 0. Conversely, x belongs to degree $(\tilde{A})$ less strongly when $\tilde{A}(x)$ is closer to 0. The membership function $\tilde{A}(x)$ is a discriminant function $\mu_A(x)$, but the fuzzy set $(\tilde{A})$ is a regular set (A) if the value region of $\tilde{A}(x)$ is $\{0, 1\}$.

While the membership function value in (0, 1) defines a distinct border, also calling distinct subsets of fuzzy sets, an element with a membership degree of 1 definitely belongs to this fuzzy set; an element with a membership degree of 0 does not definitely belong to this fuzzy set.

III. We call fuzzy sets in $f(x) \setminus C(x)$ true fuzzy sets.

4.2 | Intervals and Fuzzy Numbers

Intervals and fuzzy numbers are two mathematical notions that are used to describe inaccuracies in data or measurements. While they have certain commonalities, their uses and mathematical representation set them apart.

4.2.1 | Intervals

A range of values between two endpoints is called an interval, and it's commonly expressed as [a, b], where 'a' denotes the lower bound and 'b' denotes the upper bound.

In tasks like optimization, constraint satisfaction, and equation solving under uncertainty, intervals are frequently used in mathematics, computer science, and engineering. They are used to represent precise uncertainty, where we know the range within which the value lies but are uncertain about the exact value itself.

4.2.2 | Fuzzy numbers

When a numerical value is unclear or lacking in precision, fuzzy numbers are utilized to express it. because they describe uncertain values in a more flexible way than intervals, which reflect precise uncertainty. This allows for partial membership of a value in a collection, and a membership function that gives each value in a fuzzy number's domain a degree of membership is commonly used to describe fuzzy numbers. This membership function may be Gaussian, trapezoidal, triangular, or another type.

Fuzzy systems, fuzzy set theory, and fuzzy logic all use fuzzy numbers to deal with ambiguous data or linguistic factors.

Among its advantages is that, when precise numerical numbers are unavailable or challenging to measure, they are especially helpful in decision-making processes.

To sum up, fuzzy numbers are used to express ambiguity or vagueness in numerical values using flexible membership functions, whereas intervals are used to indicate precise ranges of values with definite endpoints. Both ideas are useful resources for handling uncertainty in many situations.

Definition 4. Let $\tilde{A}$ which was a normalized fuzzy set. $\tilde{A}$ is the trapezoidal fuzzy number and defined as follows:

$$\mu_{\tilde{A}}(x) = \mu_{\tilde{A}}(x, a_1, a_2, a_3.a_4) = \begin{cases} (a_1 - x)/(a_1 - a_2), & a_1 \le x \le a_2, \\ 1, & a_2 < x < a_3, \\ (a_4 - x)/(a_4 - a_3), & a_3 \le x \le a_4, \\ 0, & \text{otherwise.} \end{cases}$$

Hint: The trapezoidal fuzzy number becomes a triangular fuzzy number if a_2 and a_3 are equal. It is represented as (a_1, a_2, a_3).

$$\mu_{\tilde{A}}(x)=\mu_{\tilde{A}}(x,a_1,a_2,a_3)=\begin{cases}(x-a_1)/(a_2-a_1), x\in[a_1,a_2),\\ (a_3-x)/(a_3-a_2), x\in[a_2,a_3],\\ 0, \qquad \text{otherwise.}\end{cases}$$

Due to the fact that the (α-cut) of any fuzzy number is a closed interval of real numbers, (α-cut) fully and exclusively represents a fuzzy number. Because of this, we are able to describe arithmetic operations on the fuzzy number in terms of α reductions. Given two fuzzy sets $\tilde{A}=\left[A_\alpha^{LB},A_\alpha^{UB}\right]$ and $\tilde{B}=\left[B_\alpha^{LB},B_\alpha^{UB}\right]$. The following is a definition of the arithmetic operations:

I. Addition: $\left(\tilde{A}+\tilde{B}\right)=\left[A_\alpha^{LB}+B_\alpha^{LB},A_\alpha^{UB}+B_\alpha^{UB}\right]..$

II. Scalar multiplication: $\left(k\tilde{A}\right)_\alpha=\left[kA_\alpha^{LB},kA_\alpha^{UB}\right]$, if $k\rangle 0$ and $\left(k\tilde{A}\right)_\alpha=\left[kA_\alpha^{UB},kA_\alpha^{LB}\right]$, if $k\langle 0$.

III. Multiplication $\left(\tilde{A}.\tilde{B}\right)=\left[\text{Min}\left(A_\alpha^{LB}B_\alpha^{LB},A_\alpha^{LB}B_\alpha^{UB},A_\alpha^{UB}B_\alpha^{LB},A_\alpha^{UB}B_\alpha^{UB}\right),\text{Max}\left(A_\alpha^{LB}B_\alpha^{LB},A_\alpha^{LB}B_\alpha^{UB},A_\alpha^{UB}B_\alpha^{LB},A_\alpha^{UB}B_\alpha^{UB}\right)\right].$

IV. Division: $\left(\frac{\tilde{A}}{\tilde{B}}\right)_\alpha=\left[\text{Min}\left(\frac{A_\alpha^{LB}}{B_\alpha^{LB}},\frac{A_\alpha^{LB}}{B_\alpha^{UB}},\frac{A_\alpha^{UB}}{B_\alpha^{LB}},\frac{A_\alpha^{UB}}{B_\alpha^{UB}}\right),\text{Max}\left(\frac{A_\alpha^{LB}}{B_\alpha^{LB}},\frac{A_\alpha^{LB}}{B_\alpha^{UB}},\frac{A_\alpha^{UB}}{B_\alpha^{LB}},\frac{A_\alpha^{UB}}{B_\alpha^{UB}}\right)\right].$

Definition 5. According to [8], [10] Suppose that we are a fuzzy set in X and α belong to [0, 1]. $\tilde{A}$ is the crisp set $\tilde{A}_\alpha$ by $\alpha-\text{cut}$

$$[\tilde{A}_\alpha]=\{x\in X:\mu_{\tilde{A}}(x)\geq\alpha\}. \tag{8}$$

The membership function $\mu_{\tilde{A}}(x)=\mu_{\tilde{A}}(x,a_1,a_2,a_3),\mu_{\tilde{B}}(x)=\mu_{\tilde{B}}(x,b_1,b_2,b_3,b_4)$ and $\tilde{A}^{\text{Triangular}}$, $\tilde{B}^{\text{Trapezoidal}}$ be a fuzzy number, then

$$\begin{aligned}&[\tilde{A}_\alpha]=[a_1+\alpha(a_2-a_1),a_3-\alpha(a_3-a_2)],\\&[\tilde{B}_\alpha]=[b_1+\alpha(b_2-a_1),b_3+\alpha(b_4-b_3)].\end{aligned} \tag{9}$$

Definition 6 (ranking method of fuzzy numbers). If we assume $\left(\tilde{A}_1,\tilde{A}_2,\tilde{A}_3\right)$ they are fuzzy numbers with $\alpha-\text{cut}.\left[\tilde{A}_1\right]_\alpha=\left[a_{1\alpha}^-,a_{1\alpha}^+\right],\left[\tilde{A}_2\right]_\alpha=\left[a_{2\alpha}^-,a_{2\alpha}^+\right],\left[\tilde{A}_3\right]_\alpha=\left[a_{3\alpha}^-,a_{3\alpha}^+\right]$. based on $\alpha-\text{cut}$ can be used probability and necessity theories to rank fuzzy numbers as follows:

Method 1

Say $\tilde{A}_2$ greater than or equal to $\tilde{A}_1$ and denoted by $\tilde{A}_1\leq\tilde{A}_2$ if and only if $a_{1\alpha}^-\leq a_{2\alpha}^-$, and $a_{1\alpha}^+\leq a_{2\alpha}^+$ for $\alpha\in(0,1]$. in addition to, for $k_1,k_2\geq 0$ we say $k_1\tilde{A}_1+k_2\tilde{A}_2\leq\tilde{A}_3$ if and only if $k_1a_{1\alpha}^-+k_2a_{2\alpha}^-\leq a_{3\alpha}^-$, and $k_1a_{1\alpha}^++k_2a_{2\alpha}^+\leq a_{3\alpha}^+$.

Method 2

Say $\tilde{A}_2$ greater than or equal to $\tilde{A}_1$ and denoted by $\tilde{A}_1\leq\tilde{A}_2$ if and only if $a_{1\alpha}^+\leq a_{2\alpha}^+$ for $\alpha\in(0.5,1]$. in addition to, for $k_1,k_2\geq 0$ we say $k_1\tilde{A}_1+k_2\tilde{A}_2\leq\tilde{A}_3$ if and only if $k_1a_{1\alpha}^++k_2a_{2\alpha}^+\leq a_{3\alpha}^+$

Also, if we had two fuzzy sets there are numerous techniques to rank fuzzy numbers. A particularly efficient method involves a ranking function, denoted as F(R) → J, which assigns each fuzzy number to a position on the real number line, where a clear natural order exists. This approach establishes orders within F(R) as outlined below:

I. $\tilde{A}_1\geq\tilde{A}_2$ if and only if $d(\tilde{A})_1\geq d(\tilde{A}_2)$.

II. $\tilde{A}_1 \le \tilde{A}_2$ if and only if $d(\tilde{A})_1 \le d(\tilde{A}_2)$.

III. $\tilde{A}_1 = \tilde{A}_2$ if and only if $d(\tilde{A})_1 = d(\tilde{A}_2)$.

The ranking of triangular fuzzy numbers whether triangular or trapezoidal as follows:

Ranking of triangular fuzzy number

Let $\tilde{A}_1 = (a_1, a_2, a_3)$ be a fuzzy triangular number then

$$r(\tilde{A}_1) = \frac{a_1 + 4a_2 + a_3}{6}. \tag{10}$$

Ranking of trapezoidal fuzzy number

Let $\tilde{A}_2 = (a_1, a_2, a_3, a_4)$ be a fuzzy trapezoidal number then

$$r(\tilde{A}_2) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6}. \tag{11}$$

Definition 7. Presumably a trapezoidal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$, after which we'll state a triangular fuzzy number in relation to the average to this fuzzy set is $\sum_1^n \frac{\tilde{a}_i}{n} = I$, to convert a trapezoidal fuzzy number into a triangular fuzzy number,

5 | Creating a LPP Using Fuzzy

The following is a general form of a Fuzzy Number Linear Programming (FNLP) problem [11], [18], [20]:

Definition 8. The following defines a FNLP problem:

$$\begin{aligned} &\text{Maximization}(\tilde{Z}) \approx \tilde{C}X, \\ &\text{S.T} \quad \tilde{A}X \le \tilde{b}, \\ &X \ge 0, \end{aligned} \tag{12}$$

where $(\le)$ and $(\approx)$ signifies both equality and inequality concerning a certain linear ranking function R, $(\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{C}_j)$ belong to F(R) for (i = 1, 2, … , m) and (j=1, 2, … , n) where $\tilde{A} = (\tilde{a}_{ij})_{mxn}$, $\tilde{C} = (\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_n)$, $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_m)$.

Definition 9. A feasible solution is any (x) that satisfies the FNLP set of constraints. As an assumption, all possible solutions to FNLP should be represented by (Q). It represents a feasible solution. opinion that $(X^* \in Q)$ is an optimal solution for (FNLP) if $(\tilde{C}X \le \tilde{C}X^*)$ for all $(X \in Q)$.

Definition 10. We state that, for a given linear ranking function R, the fuzzy number $(\tilde{a})$ and the real number (a) correspond, if $(a = R(\tilde{a}))$ and, If there is no solution for the LP problem, then there is also no solution for the FNLP problem.

6 | Fuzzy LFP Problem

The possibility of achieving each objective goal to the highest degree in a fuzzy decision-making environment is essentially a representation of the objective goals' aspirational levels of achievement. If an imprecise target level is assigned to each of the objectives within multi-objective LFP (MOLFP), these objectives become characterized as fuzzy goals [7], [8], [15].

To convert the MOLFP problem into an LPP, a method should be implemented such that the LPP's ideal solution becomes an effective one for the MOLFP problem. Variable detours, the (Max-Min) technique, and α-cut are also used in the design of our work.

Procedure of algorithm

We will talk about the steps to reduce the FMOLFP to MOLFP model, which are explained as follows:

First procedure

Transforming the FMOLFP *Model (13)* into a MOLFP model can be achieved through a ranking approach that uses trapezoidal or triangular fuzzy numbers to characterize fuzzy coefficients.

Second procedure

For each objective function, convert each fuzzy coefficient and scalar, which are in a trapezoidal fuzzy number, to a crisp number depending on *Eqs. (10)* or *(11)* and depending on the type of fuzzy numbers under study.

Third procedure

For each constraint, convert Left-Hand Side (LHS) of each fuzzy coefficient, which is a trapezoidal or triangular fuzzy number, to a crisp number depending on *Eqs. (10)* or *(11)*.

Fourth procedure

Consider the Right-Hand Side (RHS) of each fuzzy constraint as a crisp number.

Five procedure: following Charnes and Cooper's approach [21], *Eq. (1)* is transformed into the following linear problem utilizing variable transformations, by [10], [13], [15].

The MOLFPP is often described as follows:

$$\text{Maximize}\left\{\frac{\tilde{C}^t X+\tilde{D}}{\tilde{P}^t X+\tilde{Q}}\right\},\text{, s.t } \tilde{A}X\le\tilde{b},\ \ X\ge 0, \tag{13}$$

where X=(X_1, X_2, … , X_n), the matrix is an (m*n) with fuzzy numbers, $\tilde{b}$ and $\tilde{a}_{ij}$ matrix of (m*1) with fuzzy numbers, (j=1, 2, 3, … , n), $(\tilde{b}_i = 1,2,....,m)$. The *Eq. (9)* is changed by using $\alpha-\text{cut}$ as follows:

$$\text{Maximize}\quad \frac{\left[\underline{C}^t,\overline{C}^t\right]X+\left[\underline{D},\overline{D}\right]}{\left[\underline{P}^t,\overline{P}^t\right]X+\left[\underline{Q},\overline{Q}\right]},\text{s.t}\left[\underline{A},\overline{A}\right]X\le\left[\underline{b},\overline{b}\right],X\ge 0. \tag{14}$$

The following results from changing *Eq. (14)* using operations on fuzzy intervals and number

$$\text{Maximize}\quad \tilde{F}(X)=\left[\underline{F}(X),\overline{F}(X)\right]. \tag{15}$$

s.t $S=\{\underline{A}X\le\underline{b},\overline{A}X\le\overline{b},\overline{P}^t X+\overline{Q},\underline{P}^t X+\underline{Q}>0, X\ge 0\}$,

where $\underline{F}(X)=\left\{\frac{\underline{C}^t X+\underline{D}}{\overline{P}^t X+\overline{Q}}\right\}$, if Minimize $\underline{C}^t X+\underline{D}\ge 0$ Otherwise, $\underline{F}(X)=\left\{\frac{\underline{C}^t X+\underline{D}}{\underline{P}^t X+\underline{Q}}\right\}$,

and

$\overline{F}(X)=\left\{\frac{\overline{C}^t X+\overline{D}}{\underline{P}^t X+\underline{Q}}\right\}$, if Maximize $\overline{C}^t X+\overline{D}\ge o$ Otherwise, $\overline{F}(X)=\left\{\frac{\overline{C}^t X+\overline{D}}{\overline{P}^t X+\overline{Q}}\right\}$,

S is expected to be a regular set, as well as a non-empty, bounded feasible region. We believe $\underline{F}(X)=\left\{\dfrac{\underline{C}^tX+\underline{D}}{\overline{P}^tX+\overline{Q}}\right\}$, $\overline{F}(X)=\left\{\dfrac{\overline{C}^tX+\overline{D}}{\underline{P}^tX+\underline{Q}}\right\}$ in the rest of the paper.

Eq. (8) can be expressed as follows using [10]:

$$\underset{X\in S}{\text{Maximize}}\ \left\{\underline{F}(X),\overline{F}(X)\right\}=\left\{\frac{\underline{C}^tX+\underline{D}}{\overline{P}^tX+\overline{Q}},\frac{\overline{C}^tX+\overline{D}}{\underline{P}^tX+\underline{Q}}\right\}. \tag{16}$$

To convert *Eq. (16)* into non-negative numerators and positive denominators problems should be determined by the membership functions of the objectives, and it was an equivalent bi-objective problem considered in terms of the membership functions. In reality, non-negativities conditions are used to prove that this approach produces an effective solution. For this purpose, let $\underset{X\in S}{\text{Maximize}}\,\underline{F}(X)=\underline{F}^{max}$, $\underset{X\in S}{\text{Minimize}}\,\underline{F}(X)=\underline{F}^{Min}$, $\underset{X\in S}{\text{Maximize}}\,\overline{F}(X)=\overline{F}^{max}$, $\underset{X\in S}{\text{Minimize}}\,\overline{F}(X)=\overline{F}^{Min}$.

So, the functions of membership concerning the objective functions $\underline{F}(X)$, $\overline{F}(X)$ are $\underline{\mu}(x)=\dfrac{L^tX+M}{\overline{P}^tX+\overline{Q}}$, $\overline{\mu}(x)=\dfrac{N^tX+O}{\underline{P}^tX+\underline{Q}}$, successive, where

$$\begin{aligned} &L=\left(\frac{\underline{C}}{\underline{F}^{max}-\underline{F}^{min}}-\underline{F}^{min}\,\overline{P}\right),\quad M=\left(\frac{\underline{D}}{\underline{F}^{max}-\underline{F}^{min}}-\underline{F}^{min}\,\overline{Q}\right),\ \text{all } X\in S\\ &N=\left(\frac{\overline{C}}{\overline{F}^{max}-\overline{F}^{min}}-\overline{F}^{min}\,\underline{P}\right),\quad O=\left(\frac{\overline{D}}{\overline{F}^{max}-\overline{F}^{min}}-\overline{F}^{min}\,\underline{Q}\right)\end{aligned} \tag{17}$$

Since, $\underline{\mu}(x)$, $\overline{\mu}(x)\in[0,1]$, $\underline{P}^tX+\underline{Q}$, $\overline{P}^tX+\overline{Q}\rangle 0$, then $L^tX+M, N^tX+O\geq 0$, for all $X\in S$. According to the membership functions, *Eq. (16)* takes the following form

$$\underset{X\in S}{\text{Maximize}}\left\{\underline{\mu}(x)=\frac{L^tX+M}{\overline{P}^tX+\overline{Q}},\quad \overline{\mu}(x)=\frac{N^tX+O}{\underline{P}^tX+\underline{Q}}\right\}. \tag{18}$$

Next using setting

$$\lambda=\text{Min}\left\{\frac{1}{\overline{P}^tX+\overline{Q}},\ \frac{1}{\underline{P}^tX+\underline{Q}}\right\},\lambda X=Y,\quad \text{for all } X\in S. \tag{19}$$

Can transform *Eq. (18)* into

$$\begin{aligned}&\text{Maximize}\left\{L^tY+\lambda M, N^tY+\lambda O\right\},\\ &\text{s.t}\ \Psi=\left\{\underline{A}Y-\lambda\underline{b}\leq 0,\overline{A}Y-\lambda\overline{b}\leq 0,\overline{P}^tY+\lambda\underline{Q}\leq 1,Y,\lambda\geq 0\right\}.\end{aligned} \tag{20}$$

assumed to (Ψ) should be a regular set.

Suggestion 1. In *Eq. (20)*, (λ) variable should be greater than or equal to zero.

Proof: let $(\hat{Y},0)\in\Psi$ then $\underline{A}\hat{Y}\leq 0,\overline{A}\hat{Y}\leq 0$. So, $\hat{X}\in S$ produces $\underline{A}(\hat{X}+\beta\hat{Y})=\underline{A}\hat{X}+\beta\underline{A}\hat{Y}\leq\underline{A}\hat{X}\leq 0,$, $\overline{A}(\hat{X}+\beta\hat{Y})=\overline{A}\hat{X}+\beta\overline{A}\hat{Y}\leq 0$, for all $\beta\geq 0$ feasibility of point of S is $\hat{X}+\beta\hat{Y}$, for all $\beta\geq 0$. Therefore, S must be unlimited, this is in contrast to the fact that S is a regular set.

Suggestion 2. If $(\overline{Y},\overline{\lambda})\in\Psi$, then $\dfrac{\overline{Y}}{\overline{\lambda}}\in S.$.

Proof: where $\left(\bar{Y},\bar{\lambda}\right)\in\Psi$, then $\bar{Y}\geq 0,\bar{\lambda}>0,\ \underline{A}\bar{Y}-\bar{\lambda}\underline{b},\bar{A}\bar{Y}-\bar{\lambda}\bar{b}\leq 0.$ Therefore,

$$\frac{\bar{Y}}{\bar{\lambda}}\geq 0,\underline{A}\left(\frac{\bar{Y}}{\bar{\lambda}}-\underline{b}\right)=\frac{1}{\bar{\lambda}}\left(\underline{A}\bar{Y}-\bar{\lambda}\underline{b}\right)\leq 0,\bar{A}\left(\frac{\bar{Y}}{\bar{\lambda}}\right)-\bar{b}=\frac{1}{\bar{\lambda}}\left(\bar{A}\bar{Y}-\bar{\lambda}\bar{b}\right)\leq 0.$$

Let's suppose $\beta\leq L^{t}Y+\lambda M,\beta\leq N^{t}Y+\lambda O,\text{for all }\left(Y,\lambda\right)\in\Psi$. It has been changed to

$$\text{Maximize}\quad\beta, \tag{20}$$

s.t $\Omega=\left\{\underline{A}Y-\lambda\underline{b}\leq 0,\bar{A}Y-\lambda\bar{b}\leq 0,\bar{P}^{t}Y+\lambda\bar{Q}\leq 1,\underline{P}^{t}Y+\lambda\underline{Q}\leq 1,\beta\leq L^{t}Y+\lambda M,\beta\leq N^{t}Y+\lambda O,Y,\lambda,\beta\geq 0\right\}$, here the ($\Omega$) is a regular set.

Lemma 1. From *Eq. (21)* had become the optimal solution is unique.

Proof: if it was the optimal solution is not unique based on the assumed $\left(Y^{*},\lambda^{*},\beta^{*}\right)$, that is means the constraint is active at the optimum when the $\beta\geq 0,$ i.e. $\beta^{*}=0.$ Let's look at the other words, if the $(Y,\lambda,\beta)\in\Omega$, so then $\beta=0.$ so, either $L^{num}Y+\lambda M=0$ or $N^{num}Y+\lambda O=0,\text{for all }\left(Y,\lambda,0\right)\in\Omega$. Without losing the generality, let $L^{num}Y+\lambda M=0,\text{for all }\left(Y,\lambda,0\right)\in\Omega$. Since $\lambda>0$, then $L^{num}Y+M=0,\text{for all }X\in S$; that explains $\underline{\mu}(x)=0,\text{for all }X\in S.$ This contradicts reducing the *Eq. (15)* to a single objective (LFP) problem.

Theorem 2. $X^{*}=\frac{Y^{*}}{\lambda^{*}}$ is the active solution for *Eq. (18)* with assume $\left(Y^{*},\lambda^{*},\beta^{*}\right)$ can be the optimal solution to *Eq. (21)*.

Theorem 3. The optimal solution, assuming that $\left(Y^{*},\lambda^{*},\beta^{*}\right)$ for *Eq. (21)* and $X^{*}=\frac{Y^{*}}{\lambda^{*}}$ is a viable solution for *Eq. (15)*.

Theorem 4. The optimal solution, assuming that $\left(Y^{*},\lambda^{*},\beta^{*}\right)$ for *Eq. (21)* and $X^{*}=\frac{Y^{*}}{\lambda^{*}}$ is $\varepsilon-$optimal solution for *Eq. (15)* where $\varepsilon=\text{Max}\left\{\underline{F}^{\max}-\underline{F}(X^{*}),\bar{F}^{\max}-\bar{F}(X^{*})\right\}.$

Proof: *Theorem 3* illustrates that ($X^{*}=\frac{Y^{*}}{\lambda^{*}}$) is effective for *Eq. (18)*, consequently,

Situation 1: $\underline{\mu}(X)<\underline{\mu}(X^{*}),\text{for all }X\in S.$ This means, $\underline{F}(X)<\underline{F}(X^{*}),\text{for all }X\in S.$

Allow us to :$\varepsilon_{1}=\bar{F}^{\max}-\bar{F}(X^{*})$.Hence, $\bar{F}(X)=\left[\underline{F}(X),\bar{F}(X)\right]\leq\left[\underline{F}(X^{*})+\varepsilon_{1},\bar{F}(X^{*})+\varepsilon_{1}\right]$

$=\bar{F}(X^{*})+\varepsilon_{1},\text{for all }X\in S.$

This indicates $X^{*}=\frac{Y^{*}}{\lambda^{*}}$ is an $\varepsilon_{1}-$effective solution for *Eq. (15)*.

Situation 2: $\bar{\mu}(X)<\bar{\mu}(X^{*}),\text{for all }X\in S.$As an outcome, $\bar{F}(X)<\bar{F}(X^{*}),\text{for all }X\in S.$

Allow us to:$\varepsilon_{2}=\underline{F}^{\max}-\underline{F}(X^{*})$.Hence, $\bar{F}(X)=\left[\underline{F}(X),\bar{F}(X)\right]\leq\left[\underline{F}(X^{*})+\varepsilon_{2},\bar{F}(X^{*})+\varepsilon_{2}\right].$

$=\bar{F}(X^{*})+\varepsilon_{2},\text{for all }X\in S.$ This implies, $X^{*}=\frac{Y^{*}}{\lambda^{*}}$, is an ε_{2}- effective solution for Eq. (15).

If we set, $\varepsilon=\text{Max}\left\{\varepsilon_{1},\varepsilon_{2}\right\}$, then $\bar{F}(X)=\left[\underline{F}(X),\bar{F}(X)\right]\leq\left[\underline{F}(X^{*})+\varepsilon,\bar{F}(X^{*})+\varepsilon\right]=\bar{F}(X^{*})+\varepsilon,\text{for all }X\in S.$

7 | Application Side

This solution aims to determine the best and most efficient production mix that achieves maximum profits relative to minimum costs in the presence of constraints imposed on the production process. These constraints may limit the company's ability to provide products in the required quantity and time.

7.1|Description Data and Construct Mathematical Model

Outline of the field of application:

The Light Industries Company was selected as the field of application. The company is located in Al-Zafaraniya, Baghdad, and is one of the mixed-sector companies. It was established in 1959 with a nominal capital of half a million dinars. The company has evolved over the years, and its current capital is 16.8 billion Iraqi dinars.

The company's area is approximately 283,000 square meters and includes three factories:

I. Refrigerator factory, which produces various types of refrigerators in different sizes with an area of about 30,000 square meters.

II. Oil heaters and gas cookers factory with an area of about 25,000 square meters.

III. Freezers factory, which produces various types of freezers with an area of about 20,000 square meters.

There are plans to introduce modern products and diverse models.

The focus was on two main products within the company, then building the mathematical model after obtaining data specific to the selected products.

7.2|Nomenclature

X_1: Represents the number of units produced for frozen with a size of 13 feet.

X_2: Represents the number of units produced for frozen with a size of 10 feet.

In addition to specifying the variables of the mathematical model, constants are assigned, and determining these constants refers to the fixed quantities whose values do not change. These constants represent fixed values that do not change during the execution of the mathematical model. When we do not assign a specific value to these constants, the constant can take any numerical value. In the case of not specifying a specific value, this constant is referred to as a parameter. Thus, the parameter is referred to as the variable constant that can change during the operation of the model, and the term "parameter" is used to indicate this variable constant.

Therefore, the constants and parameters of the mathematical model for this research are as follows:

C^{num} : coefficients of decision variables in the numerator objective function (which represent profits and are fuzzy numbers).

D^{den} : coefficients of decision variables in the denominator objective function (which represent costs, which are also fuzzy numbers).

X : a vector representing n-dimensional decision variables.

A : matrix of degree mxn (are fuzzy numbers).

b : A vector representing the available resources for constraints (are fuzzy numbers).

θ_1, θ_2 : is constants (are fuzzy numbers).

$$\text{Max}(\tilde{F}) = \frac{(4.731, 5.125, 5.591)X_1 + (5.846, 6.333, 6.91)X_2 + (16.154, 17.5, 19.091)}{(27.962, 30.292, 33.045)X_1 + (24.923, 27, 29.455)X_2 + (155.568, 168.533, 183.854)},$$

s.t.

$$(2.43,\ 2.64,\ 2.88)X_1 + (2.43,\ 2.64,\ 2.88)X_2 \leq (57904,\ 62730,\ 68432),$$
$$(1,\ 1,\ 1)X_1 \leq (6923,\ 7500, 8181),$$
$$(1, 1, 1)X_2 \leq (1538,\ 1666, 1818),$$
$$(1,\ 1,\ 1)X_1 \leq (923,\ 1000, 1090),$$
$$(1, 1, 1)X_2 \leq (153, 166, 181),$$
$$X_1,\ X_2 \geq 0.$$

The primary objective of this model is to determine the most efficient production mix, aiming to achieve maximum profitability while minimizing costs. It operates based on the company's existing raw materials, striving to optimize resource utilization. By focusing on maximizing profits and cost reduction, this approach aims to yield significant returns for the company.

Using *Eq. (9)* and $\alpha = 0.8$, We get

$$\text{Max} \quad \bar{F}(X) = \frac{[5.046, 5.218]X_1 + [6.236, 6.448]X_2 + [17.231, 17.818]}{[29.826, 30.843]X_1 + [26.585, 27.491]X_2 + [165.94, 171.597]},$$

s.t.

$$[2.598,\ 2.688]X_1 + [2.598,\ 2.688]X_2 \leq [61764.8,\ 63870.4],$$
$$[1,\ 1]X_1 \leq [7384.6,\ 7636.2],$$
$$[1,\ 1]X_2 \leq [1640.4, 1696.],$$
$$[1,\ 1]X_1 \leq [984.6,\ 1018],$$
$$[1,\ 1]X_2 \leq [163.4,\ 169].$$

The *Eq. (15)* is formulated as follows:

$$\underset{X \in S}{\text{Max}} = \left[\frac{5.046X_1 + 6.236X_2 + 17.231}{30.843X_1 + 27.491X_2 + 171.597}, \frac{5.218X_1 6.448X_2 + 17.818}{29.826X_1 26.585X_2 + 165.94}\right],$$

s.t.

$$2.598X_1 + 2.598X_2 \leq 61764.8,$$
$$X_1 \leq 7384.6,$$
$$X_2 \leq 1640.4,$$
$$X_1 \leq 984.6,$$
$$X_2 \leq 163.4,$$
$$2.688X_1 + 2.688X_2 \leq 63870.4,$$
$$X_1 \leq 7636.2,$$
$$X_2 \leq 1696.4,$$
$$X_1 \leq 1018,$$
$$X_2 \leq 169.$$

The *Eq. (16)* is then formulated as follows:

$$\text{Max}\quad \left\{\underline{F}(X),\overline{F}(X)\right\}=\left\{\frac{5.046Y_1+6.236Y_2+17.231}{30.843Y_1+27.491Y_2+171.597},\frac{5.218Y_1+6.448Y_2+17.818}{29.826Y_1+26.585Y_2+165.94}\right\},$$

s.t.

$2.598Y_1 + 2.598Y_2 - 61764.8\lambda \le 0,$

$Y1 - 7384.6\lambda \le 0,$

$Y2 - 1640.4\lambda \le 0,$

$Y1 - 984.6\lambda \le 0,$

$Y2 - 163.4\lambda \le 0,$

$2.688Y_1 + 2.688Y_2 - 63870.4\lambda \le 0,$

$Y1 - 7636.2\lambda\lambda \le 0,$

$Y2 - 1696.4\lambda \le 0,$

$Y1 - 1018\lambda \le 0,$

$Y2 - 169\lambda \le 0.$

By using Charnes and Cooper rule, We find maxima and minima as follows: $\underline{F}^{Max}=0.222172, \underline{F}^{Min}=0, \overline{F}^{Max}=0.237746, \overline{F}^{Min}=0$. And then define the membership functions and *Eq. (17)* are follows:

$L = 22.69693164X_1 + 28.04955722X_2 \quad , \quad M = 77.50511873,$

$N = 21.9477561X_1 + 27.12133601X_2 \quad , \quad O = 74.94540401,$

$$\underline{\mu}(X)=\frac{22.69693164X_1+28.04955722X_2+77.50511873}{30.843X_1+27.491X_2+171.597},$$

$$\overline{\mu}(X)=\frac{21.9477561X_1+27.12133601X_2+74.94540401}{29.826X_1+26.585X_2+165.94}.$$

The formula for *Eq. (19)* is: $\lambda = \text{Min}\left\{\frac{1}{30.843X_1+27.491X_2+171.597},\frac{1}{29.826X_1+26.585X_2+165.94}\right\}, Y=\lambda X.$

$$\text{Maximize}\quad \left\{22.69693164X_1+28.04955722X_2+77.50511873\lambda,\quad 21.9477561X_1+27.12133601X_2+74.94540401\lambda\right\},$$

s.t.

$2.598Y_1 + 2.598Y_2 - 61764.8\lambda \le 0,$

$Y_1 - 7384.6\lambda \le 0,$

$Y_2 - 1640.4\lambda \le 0,$

$Y_1 - 984.6\lambda \le 0,$

$Y_2 - 163.4\lambda \le 0,$

$30.843Y_1 + 27.491Y_2 - 171.597\lambda \le 1,$

$2.688Y_1 + 2.688Y_2 - 63870.4\lambda \le 0,$

$Y_1 - 7636.2\lambda \le 0$

$Y_2 - 1696.4\lambda \le 0$

$Y_1 - 1018\lambda \le 0$

$Y_2 - 169\lambda \le 0$

$29.826Y_1 + 26.585Y_2 - 165.94\lambda \le 1.$

The result of *Eq. (20)* is as follows:

$\text{Maximize}\quad \beta,$

s.t

$$\Omega = \Psi \cup \{\beta \le 22.69693164X_1 + 28.04955722X_2 + 77.50511873\lambda, \beta \le 21.9477561X_1 + 27.12133601X_2 + 74.94540401\lambda\},$$

For the main problem, the optimal solution is $\left(Y^*, \lambda^*, \beta^*\right) = \left(Y_1^*, Y_2^*, \lambda^*, \beta^*\right) = (0, 0.03627592, 0.00021465)$.

$$X_1^* = \frac{0}{0.00021465} = 0, X_2^* = \frac{0.03627592}{0.00021465} = 169,$$

$$X^* = \frac{Y^*}{\lambda^*} = (0, \quad 169),$$

$$\underline{F}(X^*) = \frac{6.2356(169) + 17.2308}{27.491(169) + 171.5972} = 0.2223208,$$

$$\bar{F}(X^*) = \frac{6.4484(169) + 17.8182}{26.5846(169) + 165.94} = 0.2377464,$$

$$\underline{F}(X^*) = 0.2223208, \quad \bar{F}(X^*) = 0.2377464, \quad \tilde{F}(X^*) = [0.2223208, \quad 0.2377464],$$

$$\varepsilon = \mathrm{Max}\left\{\underline{F}^{\max} - \underline{F}(X^*), \quad \bar{F}^{\max} - \bar{F}(X^*)\right\} = \mathrm{Max}\{-0.0001486323, \quad 0\} = 0,$$

$$\tilde{F}(X) = \left[\underline{F}(X), \bar{F}(X)\right] \le \left[\underline{F}(X^*) + \varepsilon, \quad \bar{F}(X^*) + \varepsilon\right] = \tilde{F}(X^*) + \varepsilon, \text{ for all } X \in S.$$

Here are the feasible region S's extreme points. By numbers, we observe that

$\underline{F}(X^{EP}) + \varepsilon < \underline{F}(X^*) + \varepsilon, \quad \bar{F}(X^{EP}) < \bar{F}(X^*)$, where it is believed that X^{EP} is an extreme point, thus, S's convexity, along with pseudo-convexity of $\underline{F}(X)$ and $\bar{F}(X)$ implies that $\underline{F}(X) < \underline{F}(X^*) + \varepsilon, \bar{F}(X) < \bar{F}(X^*) + \varepsilon, \text{for all } X \in S$. Consequently, $\tilde{F}(X) = \left[\underline{F}(X), \bar{F}(X)\right] \le \left[\underline{F}(X^*) + \varepsilon = 0.2223208, \quad \bar{F}(X^*) + \varepsilon = 0.2377464\right] = \tilde{F}(X^*) + \varepsilon, \text{ for all } X \in S.$

This indicates (X^*) is the main fuzzy problem's ϵ-optimal solution. This indicates (X^*) is the unique optimal solution to the primary fuzzy problem.

Table 1. Extreme points of S_α=0.8 and their values of $\tilde{F}(X)$.

Extreme Point	$\tilde{F} = \left[\underline{F}(X^{EP}), \quad \bar{F}(X^{EP})\right]$
(50, 90)	[0.212, 0.198]
(50, 100)	[0.213, 0.200]
(25, 75)	[0.217, 0.203]
(30, 90)	[0.218, 0.204]
(30, 110)	[0.221, 0.207]
(0, 169) X^*	[0.2223208, 0.2377464]

8 | Discussion

When it comes to optimisation, an "extreme point" is any point in the feasible region of a problem where the objective function reaches its maximum or minimum value. The optimal solution at an extreme point denotes that it offers the best value for the objective function while satisfying all of the problem's constraints; however, it's important to keep in mind that, in some situations particularly in nonlinear optimisation problems or when working with non-convex feasible regions the optimal solution may occur at a point that is not an extreme point.

Here, were the extreme points (0, 169), that provided the best value for the objective function that would achieve the best results for the company.

9 | Conclusion

The popular approaches to solving fuzzy fractional programming rely on the application of certain equations to transform the mysterious number into an integer, our approach is predicated on transforming the fuzzy number into an interval, which serves as the foundation for a constructing a mathematical model, And use some complex equations to extract results.

The fuzzy problem was eventually transformed into an LPP. It has been demonstrated that the LPP the generated solution is an ϵ-optimal solution to the major problem. Our methodology was built using the membership function,and also depending on the the Max-Min technique, and concept of α -cut, And the transformations that get the variables. This paper covers the LFPP with any type of fuzzy numbers.

According to information from the Iraqi Light Industries Company, this multi-objective LFP problem with fuzzy coefficients was resolved, yielding, which provides a unique optimal solution for the primary fuzzy problem and obtaining useful results for DMs through extreme points.

In summary, we conclude that the approach we used is adaptable to multi-objective MOLFPP problems with fuzzy coefficients.

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability

All data are included in the text.

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References

[1] Das, S. K., Edalatpanah, S. A., & Mandal, T. (2019). A new method for solving linear fractional programming problem with absolute value functions. *International journal of operational research*, *36*(4), 455–466. https://doi.org/10.1504/IJOR.2019.104051

[2] Gurmu, E. D., & Fikadu, T. T. (2020). Fuzzy programming approach to Bi-level linear programming problems. *Journal of fuzzy extension and applications*, *1*(4), 252–271. https://doi.org/10.22105/jfea.2020.262326.1069

[3] Das, S. K., & Edalatpanah, S. A. (2022). Optimal solution of neutrosophic linear fractional programming problems with mixed constraints. *Soft computing*, *26*(17), 8699–8707. https://doi.org/10.1007/s00500-022-07171-z%0A%0A

[4] Borza, M., Rambely, A. S., & Edalatpanah, S. A. (2023). A linearization to the multi-objective linear plus linear fractional program. In *Operations research forum* (Vol. 4, p. 82). Springer.

[5] Nasseri, S. H., & Bavandi, S. (2018). Fuzzy stochastic linear fractional programming based on fuzzy mathematical programming. *Fuzzy information and engineering*, *10*(3), 324–338. https://doi.org/10.1080/16168658.2019.1612605

[6] Pramy, F. A. (2018). An approach for solving fuzzy multi-objective linear fractional programming problems. *International journal of mathematical, engineering and management sciences*, *3*(3), 280. https://www.researchgate.net/profile/Farhana-Pramy/publication/323251703_An_Approach_for_Solving_Fuzzy_Multi-Objective_Linear_Fractional_Programming_Problems/links/5a8947ce458515b8af928093/An-Approach-for-Solving-Fuzzy-Multi-Objective-Linear-Fractional-Programming-Problems.pdf

[7] Borza, M., & Rambely, A. S. (2021). A new method to solve multi-objective linear fractional problems. *Fuzzy information and engineering*, *13*(3), 323–334. https://doi.org/10.1080/16168658.2021.1938868

[8] Borza, M., & Rambely, A. S. (2022). An approach based on -cuts and max-min technique to linear fractional programming with fuzzy coefficients. *Iranian journal of fuzzy systems*, *19*(1), 153–169. https://doi.org/10.22111/ijfs.2022.6558

[9] Chakraborty, M., & Gupta, S. (2002). Fuzzy mathematical programming for multi objective linear fractional programming problem. *Fuzzy sets and systems*, *125*(3), 335–342. https://doi.org/10.1016/S0165-0114(01)00060-4

[10] Nasseri, S. H., & Bavandi, S. (2017). A suggested approach for stochastic interval-valued linear fractional programming problem. *International journal of applied operations research*, *7*(1), 23–31. http://ijorlu.liau.ac.ir/article-1-545-fa.html

[11] Das, S. K., Mandal, T., & Edalatpanah, S. A. (2017). A new approach for solving fully fuzzy linear fractional programming problems using the multi-objective linear programming. *RAIRO-operations research*, *51*(1), 285–297. https://doi.org/10.1051/ro/2016022

[12] Nachammai, A., & Thangaraj, P. (2012). Solving fuzzy linear fractional programming problem using metric distance ranking. *Applied mathematical sciences*, *6*(26), 1275–1285.

[13] Deb, M., & De, P. K. (2014). A note on fuzzy multi-objective linear fractional programming problem. *International journal of computer science and network*, *3*, 2277–5420.

[14] Ozkok, B. A. (2020). An iterative algorithm to solve a linear fractional programming problem. *Computers & industrial engineering*, *140*, 106234. https://doi.org/10.1016/j.cie.2019.106234

[15] De, P. K., & Deb, M. (2015). Solution of multi objective linear fractional programming problem by Taylor series approach. *2015 international conference on man and machine interfacing (MAMI)* (pp. 1-5). IEEE.

[16] Alharbi, M. G., & Khalifa, H. A. (2021). On solutions of fully fuzzy linear fractional programming problems using close interval approximation for normalized heptagonal fuzzy numbers. *Applied mathematics and information sciences*, *15*(4), 471–477. http://dx.doi.org/10.18576/amis/150409

[17] Ammar, E.-S., & Muamer, M. (2016). Algorithm for solving multi objective linear fractional programming problem with fuzzy rough coefficients. *Open science journal of mathematics and application*, *4*(1), 1–8.

[18] Das, S. K., & Mandal, T. (2017). A new model for solving fuzzy linear fractional programming problem with ranking function. *Journal of applied research on industrial engineering*, *4*(2), 89–96. https://doi.org/10.22105/jarie.2017.48543

[19] Nasseri, S. H., Ebrahimnejad, A., Cao, B. Y., Nasseri, S. H., Ebrahimnejad, A., & Cao, B. Y. (2019). *Fuzzy linear programming*. Springer.

[20] Das, S. K., Edalatpanah, S. A., & Mandal, T. (2020). Application of linear fractional programming problem with fuzzy nature in industry sector. *Filomat*, *34*(15), 5073–5084. https://doi.org/10.2298/FIL2015073D

[21] Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European journal of operational research*, *2*(6), 429–444.